International Journal of Theoretical Physics, Vol. 17, No. 1 (1978), pp. 5–20

Gravitational Radiation in Asymptotic de Sitter Space

Larry L. Smalley

The University of Alabama in Huntsville, Huntsville, Alabama 35807

Received March 13, 1978

A solution of the gravitational field equations is found by using an axially symmetric metric which is asymptotically a de Sitter space metric. We use the general approach of Bondi, van der Burg, and Metzner as applied to the asymptotic flat-space case and search for the necessary conditions for gravitational radiation in asymptotic de Sitter space. We find that the character of the gravitational radiation, if it exists at all, is considerably different from that obtained in the case of asymptotic flat space.

1. INTRODUCTION

The investigation and analysis of gravitational radiation in different gravitational theories and in conventional theories with alternate boundary conditions have received indirectly a new impetus from the parametrized post-Newtonian (PPN) analyses of gravitational theories (Will, 1974a). Although an investigation of gravitational radiation seems somewhat remote from the PPN analysis of gravitational theories, its importance as a definitive test between theories is just now being recognized. Part of the reason for this budding importance comes from a partial failure at the level of the PPN approximation to distinguish between alternate and viable theories of gravitation (Will, 1974b). One should point out that recent experiments tend to favor the Einsteinian theory over alternate macroscopic gravitational theory (Shapiro et al., 1976; Williams et al., 1976). Nevertheless, attempts to firmly root Einstein's general theory to experimental results has resulted in a plethora of theories that agree experimentally in the PPN approximation. Deviations between some of the theories are expected only in the "post" PPN approximation (Lee and Lightman, 1973). According to Will (1974a) it is in these higher orders that one begins to expect contributions to gravitational radiation.

0020-7748/78/0100-0005$05.00/0

In general, the PPN approximation is an expansion of the metric (Lagrangian) to fourth order in the velocity which is small in the neighborhood of the solar system. In fact, near the sun one expects that the quantities

$$\rho(x) \sim v^2 \sim U(x) \sim \frac{p(x)}{\rho}$$

where ρ is the density, U is the potential, and p is the pressure, are all of the same order of magnitude. Far from the solar system, we expect the metric to take its Minkowskiian form: $g_{\mu\nu} = (-1, 1, 1, 1)$. (Greek symbols represent the space–time components 0, 1, 2, 3, whereas Latin symbols represent spatial components 1, 2, 3.) Thus if one carried out a consistent approximation scheme to successively higher orders, gravitational radiation effects would occur in the seventh order (Will, 1974c). At this point we must ask the question: to what order is the PPN approximation still valid (i.e., does not diverge). At present, this remains an open question. Thus we are forced to check directly the contributions from gravitational radiation. A clear understanding of gravitational radiation seems necessary in order to interpret such an expansion.

A general unsolved problem exists with the understanding of gravitational radiation in the interpretation of coordinates in relativistic theories. Although the covariance of the equations seems to indicate an independence of the choice of coordinates in calculations, it does make a difference when one obtains the "physical components" necessary for comparison with experiment (Truesdell, 1953). For instance, the formulation of gravitational radiation in one frame of reference (theory) may parrot the properties of gravitational radiation, but in the end, fail the test of coordinate independence. This might occur if, for instance, a spacelike coordinate could become timelike and vice versa as inside the horizon of a blackhole (Misner, Thorne, and Wheeler, 1973).

In order to shed some light on these questions, we have investigated the gravitational radiation conditions in an asymptotic de Sitter space, a theory conformally equivalent to general relativity. Physically, this is an interesting problem for at least two reasons:

(i) The results will help demonstrate the coordinate independence of gravitational radiation, i.e., a physical interpretation useful in the detection of gravitational radiation.
(ii) It leads directly to an investigation of the group structure imposed on space-time by an asymptotic de Sitter universe.

The second case has its importance in the possible classification schemes for elementary particles (cf. Aghassi, Roman, and Santille, 1970; Tait and

Cornwell, 1971). Ironically, the importance of de Sitter space considerations may be in the small, i.e., for microscopic theories of gravitation in which one considers the gauge formalism of gravitation with local de Sitter invariance (Hsu, 1977). The former is a necessary step towards correctly quantizing the gravitational field (Sachs, 1962; Pirani, 1964).

Gravitational radiation in asymptotic spaces has been investigated by many, notably the works of Bondi, Van der Burg and Metzner (BVM) (1962) in an empty, asymptotically flat space (i.e., asymptotically Minkowskiian) and of Hawking in a dust-filled, asymptotic conformally flat Friedmann universe with negative curvature (Hawking, 1968). Here we investigate a matterless but asymptotic de Sitter universe. For completeness, we include the effect of the cosmological constant. We use the method of the BVM empty-space approach, which is well suited for this type of investigation.

In Section 2 we develop the Ricci tensor in de Sitter space. In Section 3 the de Sitter space field equations are developed in terms of a "natural" set of coordinates and then are solved in Section 4 by series expansion. In Section 5 we discuss the asymptotic de Sitter space constraints on our solution and give our conclusions in Section 6.

2. RICCI TENSOR IN DE SITTER SPACE

To use the BVM method, it is necessary to transform their metric into a de Sitter space metric. This can be done through a conformal transformation of the metric (Eisenhart, 1926; Gürsey, 1964)

$$g'_{\alpha\beta} = e^{2\sigma} g_{\alpha\beta} \tag{2.1}$$

where

$$\sigma = \ln \frac{4R^2}{4R^2 - u^2 - 2ur} \tag{2.2}$$

where R is the "radius" of de Sitter space, $u \equiv t - r$ is the null coordinate, and r is the ordinary radial coordinate. Since the BVM metric is axially symmetric, so will the de Sitter space metric be axially symmetric. Following the method of BVM, we obtain the main field equations[1]

$$\bar{R}_{11} = -\frac{4[\beta_1 - (1/2)r\gamma_1{}^2]}{r} + 2\sigma_{11} - 4\beta_1\sigma_1 - 2\sigma_1{}^2 \tag{2.3}$$

[1] The following misprints or omissions occur in Bondi et al. (1962). In the list of three index symbols: Γ^{o}_{oo} let $e^{2(\beta-\gamma)} \to e^{2(\gamma-\beta)}$, Γ^{2}_{33} let $e^{-2(\gamma-\beta)} \to e^{-2(\gamma+\beta)}$, and the correct order of expansions for $\Gamma^{3}_{o3} = -(c_o/r) - (C_o - \frac{1}{2}c^2 c_o)/r^3$, $\Gamma^{1}_{22} = r(c_o - 1) + (2M - c_{22} - 2c_2 \cot\theta + 2c\cot^2\theta + 2cc_o)$. In the supplementary condition for R_{oo} replace the term $\beta_2 UV_1/r \to \beta_2 U_1 V/r$. The constant of integration in equation (3.21) is $N + \frac{5}{6}cc_2 + \frac{2}{3}c^2 \cot\theta$ instead of the N discussed in the text. The term $8c(3cc_2)$ should be replaced by $8c(3cc_2 \cot\theta)$ in equation (3.22).

$$-2r^2\bar{R}_{12} = (r^4e^{2(\gamma-\beta)}U_1)_1 - 2r^2\left(\beta_{12} - \gamma_{12} + 2\gamma_1\gamma_2 - \frac{2\beta_2}{r} - 2\gamma_1 \cot\theta\right) + 4r^2(\beta_2 + \tfrac{1}{2}r^2e^{2(\gamma-\beta)}U_1)\sigma_1 \tag{2.4}$$

$$\begin{aligned}\bar{R}_{22}e^{2(\beta-\gamma)} + \bar{R}_{33}\frac{e^{2(\beta+\gamma)}}{\sin^2\theta} = {} & 2V_1 + \tfrac{1}{2}r^4e^{2(\gamma-\beta)}U_1{}^2 - r^2U_{12} - 4rU_2 \\ & - r^2U_1\cot\theta - 4rU\cot\theta + 2e^{2(\beta-\gamma)} \\ & \times [-1 - (3\gamma_2 - \beta_2)\cot\theta - \gamma_{22} + \beta_{22} + \beta_2{}^2 \\ & \quad + 2\gamma_2(\gamma_2 - \beta_2)] - 8r\sigma_0 \\ & + (2rV_1 + 6V - 4r^2U_2 - 4r^2U\cot\theta)\sigma_1 \\ & + 6rV\sigma_{11} - 4r^2\sigma_{10} - 8r^2\sigma_o\sigma_1\end{aligned} \tag{2.5}$$

$$\begin{aligned}\frac{e^{2(\gamma+\beta)}}{\sin^2\theta}\bar{R}_{33} = {} & 2r(r\gamma_o)_1 + (1 - r\gamma_1)V_1 - (r\gamma_{11} + \gamma_1)V - r(1 - r\gamma_1)U_2 \\ & - r^2(\cot\theta - \gamma_2)U_1 + r(2r\gamma_{12} + 2\gamma_2 + r\gamma_1\cot\theta - 3\cot\theta)U \\ & \times e^{2(\beta-\gamma)}[-1 - (3\gamma_2 - 2\beta_2)\cot\theta - \gamma_{22} + 2\gamma_2(\gamma_2 - \beta_2)] \\ & - \sigma_o(4r - 2r^2\gamma_1) + 3Vr\sigma_{11} - 2r^2\sigma_{o1} - 4r^2\sigma_o\sigma_1 \\ & - \sigma_1(-2r^2\gamma_o + 3r^2U\cot\theta - 2r^2\gamma_2U - 3V \\ & + 2r\gamma_1V - rV_1 + r^2U_2)\end{aligned} \tag{2.6}$$

The supplementary field equations become

$$\begin{aligned}\bar{R}_{o2} = {} & \beta_{o2} - \gamma_{o2} + 2\gamma_o\gamma_2 - 2\gamma_o\cot\theta - U(\beta_{22} + 2\beta_2{}^2 - 2\beta_2\gamma_2 + \beta_2\cot\theta) \\ & - \frac{V_{12}}{2r} + \frac{V_2}{2r^2} + (\gamma_1 - \beta_1)\frac{V_2}{r} + r^2e^{2(\gamma-\beta)}\left[\frac{3}{2}UU_{12} + \frac{3UU_2}{r}\right. \\ & + 2U\left(\gamma_{o1} + \frac{\gamma_o}{r}\right) + \tfrac{1}{2}U_{o1} + 2\gamma_{12}U^2 + (\gamma_o - \beta_o)U_1 + \gamma_1UU_2 \\ & + (2\gamma_2 - \beta_2)UU_1 + U_1U_2 - \frac{U_{11}V}{2r} - \frac{UV_1 + 2U_1V}{r^2} \\ & - \frac{\gamma_{11}UV + (\gamma_1 - \beta_1)U_1V + \gamma_1UV_1}{r} - \frac{\gamma_1UV}{r^2} + \frac{2\gamma_2U^2}{r} \\ & \left. + U\left(\frac{U_1}{2} + \frac{U}{r} + \gamma_1U\right)\cot\theta\right] - \tfrac{1}{2}r^4e^{4(\gamma-\beta)}UU_1{}^2 \\ & - 2\sigma_o\left[\beta_2 - r^2e^{2(\gamma-\beta)}\left(\frac{U_1}{2} + \frac{U}{r} + \gamma_1U\right)\right] \\ & - 2\sigma_1\left\{\frac{V_2}{2r} + r^2e^{2(\gamma-\beta)}\left[U\left(\frac{V}{r^2} + \frac{\gamma_1V}{r} - \gamma_o - U_2 - \gamma_2U\right) + \frac{U_1V}{2r}\right]\right\} \\ & + Ue^{2(\gamma-\beta)}[-3Vr\sigma_1{}^2 + 2r^2\sigma_{1o} + 4r^2\sigma_o\sigma_1 + 2r\sigma_o \\ & + \sigma_1(-rV_1 - V + r^2U_2 + r^2U\cot\theta)]\end{aligned} \tag{2.7}$$

$$
\begin{aligned}
\bar{R}_{oo} = {} & \frac{2\beta_{o1}V}{r} - \frac{VV_{11}}{2r^2} - \frac{\beta_{11}V^2}{r^2} - \frac{\beta_1 V^2}{r^3} - \frac{\beta_1 VV_1}{r^2} - \frac{V_o - 2\beta_o V}{r^2} \\
& + \frac{2\beta_{12}UV + \beta_2 U_1 V + \beta_1 U_2 V + 2\beta_1 UV_2}{r} + \frac{2\beta_2 UV}{r^2} - \frac{U_2 V}{2r^2} + \frac{U_2 V_1}{2r} \\
& - \frac{2UV_2}{r^2} - \frac{U_1 V_2}{2r} - \frac{2\gamma_1 UV_2}{r} - 2\beta_{o2}U - 2\beta_o U_2 + 2\gamma_{o2}U + 2\gamma_o U_2 \\
& + U_{o2} + UU_{22} + U_2{}^2 + 2(\gamma_2 - \beta_2)UU_2 + \frac{UV_{12}}{r} \\
& + (2\beta_2{}^2 - 2\beta_2\gamma_2 + \gamma_{22})U^2 + 2\gamma_o{}^2 \\
& - \cot\theta\left(2\beta_o U - 2\gamma_o U - U_o - UU_2 - \gamma_2 U^2 + \frac{UV}{2r^2} - \frac{UV_1}{2r} - \frac{\beta_1 UV}{r}\right) \\
& + r^2 e^{2(\gamma-\beta)}\left[-UU_{o1} - 2\left(\gamma_{o1} + \frac{\gamma_o}{r}\right)U^2 - 2(\gamma_o - \beta_o)UU_1 - 2U^2U_{12}\right. \\
& - 2UU_1U_2 - 2\gamma_{12}U^3 - \frac{2}{r}\gamma_2 U^3 - 3\gamma_2 U^2 U_1 + 2\beta_2 U^2 U_1 + \frac{UU_{11}V}{r} \\
& + \frac{4UU_1V}{r^2} + 2(\gamma_1 - \beta_1)\frac{UU_1V}{r} + \frac{\gamma_{11}U^2V}{r} + \frac{\gamma_1 U^2 V_1}{r} + \frac{\gamma_1 U^2 V}{r^2} \\
& \left. - \frac{3U^2U_2}{r} - \gamma_1 U^2 U_2 + \frac{U^2V_1}{r^2} + \frac{U_1{}^2 V}{2r} - U^2\left(U_1 + \frac{U}{r} + \gamma_1 U\right)\cot\theta\right] \\
& + \tfrac{1}{2} r^4 e^{4(\gamma-\beta)} U^2 U_1{}^2 - \frac{1}{2r^3} e^{2(\beta-\gamma)}[V_{22} + 2\beta_{22}V + (2\beta_2 + 2\gamma_2 \\
& + \cot\theta)(V_2 + 2\beta_2 V)] + \frac{4\phi}{B} - 2\sigma_o\left(2\beta_0 + \frac{V}{2r^2} - \frac{V_1}{2r} - \frac{\beta_1 V}{r}\right) \\
& - 2\sigma_o r^2 e^{2(\gamma-\beta)} U\left(U_1 + \frac{U}{r} + \gamma_1 U\right) - 2\sigma_1\left\{\frac{V_0}{2r} - \frac{\beta_o V}{r} - \frac{V^2}{2r^3} + \frac{VV_1}{2r^2}\right. \\
& + \frac{\beta_1 V^2}{r^2} - \frac{UV_2}{2r} - \frac{\beta_2 UV}{r} + r^2 e^{2(\gamma-\beta)}\left[-\frac{UU_1V}{r}\right. \\
& \left.\left. + U^2\left(U_2 + \gamma_2 U - \frac{V}{r^2} - \frac{\gamma_1 V}{r} + \gamma_o\right)\right]\right\} + \left(\frac{Ve^{2\beta}}{r} - r^2U^2e^{2\gamma}\right)\frac{e^{-2\beta}}{r^2} \\
& \times [-3Vr\sigma_1{}^2 + 2r^2\sigma_{1o} + 4r^2\sigma_o\sigma_1 + 2r\sigma_o \\
& + \sigma_1(-rV_1 + r^2U_2 - V + r^2U\cot\theta)]
\end{aligned}
\tag{2.8}
$$

where

$$\phi = e^\sigma = B/(B - \mu^2 - 2\mu r) \tag{2.9}$$

and

$$B = 4R^2 \tag{2.10}$$

For completeness we list the consistency field equation

$$\begin{aligned}\bar{R}_{o1} = {} & 2\beta_{10} - \frac{V_{11}}{2r} - \frac{\beta_{11}V}{r} - \frac{\beta_1 V_1}{r} - \frac{\beta_1 V}{r^2} + \beta_{12}U + \beta_2 U_1 \\ & + \frac{2\beta_2 U}{r} + \frac{U_{12}}{2} + \frac{U_2}{r} + \gamma_{12}U + \gamma_1 U_2 + 2\gamma_1\gamma_o \\ & + \left(\frac{U_1}{2} + \frac{U}{r} + \gamma_1 U\right)\cot\theta - \frac{e^{2(\beta-\gamma)}}{r^2}[\beta_{22} + 2\beta_2(\beta_2 - \gamma_2) + \beta_2\cot\theta] \\ & + r^2 e^{2(\gamma-\beta)}\left(\frac{UU_{11}}{2} + \frac{U_1^2}{2} + \frac{2UU_1}{r} + \gamma_1 UU_1 - \beta_1 UU_1\right) \\ & + 4\sigma_{o1} + 2\sigma_1\sigma_o + \frac{2\sigma_o}{r} - \frac{3V}{r}\sigma_1^2 + \sigma_1(U_2 + U\cot\theta) \\ & - \sigma_1\left(\frac{2V_1}{r} + \frac{2\beta_1 V}{r} - 2\beta_2 U - r^2 e^{2(\gamma-\beta)}UU_1\right)\end{aligned} \tag{2.11}$$

The origin of the names for the various field equations comes from the way in which a solution for the functions γ, β, U, and V occur. An expansion is assumed for γ at some instant in retarded time u. Equations (2.3)–(2.5) then yield solutions for β, U, and V at the same time. Equation (2.6) then gives the time development of γ from which β, U, and V can be found for all time. Equations (2.7) and (2.8) then represent relations between the expansion parameters and initial conditions (integration constants), and finally equation (2.11) must be trivially satisfied provided the solution for γ, β, U, and V is correct.

3. DE SITTER SPACE FIELD EQUATIONS

Since we are working in a cosmological space, in particular de Sitter space, the field equations take the generalized form (Weinberg, 1972)

$$\bar{R}_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{R} + \lambda\bar{g}_{\mu\nu} = \bar{T}_{\mu\nu} \tag{3.1}$$

where λ is the cosmological constant and $\bar{T}_{\mu\nu}$ is the energy-momentum tensor. In empty space, they become

$$\bar{R}_{\mu\nu} = \lambda\bar{g}_{\mu\nu} \tag{3.2}$$

so that, in general, the Ricci tensors of Section 2 are not set equal to zero as in the BVM case. In order to see how the field equations are solved, we note that the metric is now

$$\bar{g}_{\mu\nu} = \phi^2 g_{\mu\nu} = \begin{bmatrix} \frac{\phi^2 V e^{2\beta}}{r} - r^2\phi^2 U^2 e^{2\gamma} & \phi^2 e^{2\beta} & r^2\phi^2 U e^{2\gamma} & 0 \\ \phi^2 e^{2\beta} & 0 & 0 & 0 \\ r^2\phi^2 U e^{2\gamma} & 0 & -r^2\phi^2 e^{2\gamma} & 0 \\ 0 & 0 & 0 & -r^2\phi^2 e^{-2\gamma}\sin^2\theta \end{bmatrix} \tag{3.3}$$

Since $\bar{g}_{11} = 0$, then

$$\bar{R}_{11} = \lambda \bar{g}_{11} = 0 \tag{3.4}$$

Thus (2.3) becomes

$$0 = -\frac{4[\beta_1 - (1/2) r \gamma_1{}^2]}{r} + 2\sigma_{11} - 4\beta_1\sigma_1 - 2\sigma_1{}^2 \tag{3.5}$$

From the definition of $\sigma = \ln \phi$, then

$$\sigma_1 = \frac{2\mu}{B}\phi \tag{3.6}$$

and then

$$\sigma_{11} = \sigma_1{}^2 \tag{3.7}$$

Also

$$1 + r\sigma_1 = \frac{B - u^2}{B}\phi \tag{3.8}$$

Then (3.5) now becomes

$$\begin{aligned} 0 &= \beta_1(1 + r\sigma_1) - \tfrac{1}{2} r\gamma_1{}^2 \\ &= \frac{B - u^2}{B}\phi\beta_1 - \tfrac{1}{2} r\gamma_1{}^2 \end{aligned} \tag{3.9}$$

In terms of the variables

$$\begin{aligned} q &= r\phi \\ u' &= u \end{aligned} \tag{3.10}$$

(3.9) finally becomes (after dropping primes)

$$0 = \beta_1 q_1 \left(\frac{B - u^2}{B}\right)\phi - \tfrac{1}{2} r\gamma_1{}^2 q_1{}^2 \tag{3.11}$$

where derivatives are with respect to q except for the function

$$q_1 \equiv \frac{\partial q}{\partial r} = \frac{B - u^2}{B}\phi^2 \tag{3.12}$$

which occurs in the change of variables

$$\frac{\partial}{\partial r} = q_1 \frac{\partial}{\partial q} \tag{3.13}$$

After using (3.12) in (3.11), we get finally

$$\beta_1 = \tfrac{1}{2} q {\gamma_1}^2 \tag{3.14}$$

It should be noted that this equation is identical in form to the corresponding BVM equation except that it is now a function of the new coordinates q and u. The formal simplicity in comparison with the BVM case has directly motivated our choice of "natural" de Sitter space coordinates. Since $\bar{R}_{12} = \lambda \bar{g}_{12} = 0$, then in terms of the variables q and u, equation (2.4) now becomes

$$\left(\frac{B - u^2}{B}\right)(q^4 e^{2(\gamma-\beta)} U_1)_1 = 2q^2\gamma(2\gamma_1\gamma_2 - \gamma_{12} - 2\gamma_1 \cot\theta) + 2q^4\left(\frac{\beta_2}{q^2}\right)_1 \tag{3.15}$$

Since

$$\bar{R}_{22} e^{2(\beta-\gamma)} + \bar{R}_{33} \frac{e^{2(\gamma+\beta)}}{\sin^2\theta} = -2q^2 e^{2\beta}\lambda \tag{3.16}$$

then (2.5) becomes

$$\begin{aligned}
-2q^2 e^{2\beta}\lambda = {} & 2\left(\frac{B - u^2}{B}\right)^2 (\phi^3 V)_1 + \frac{1}{2} q^4 \left(\frac{B - u^2}{B}\right)^2 e^{2(\gamma-\beta)} {U_1}^2 \\
& - \frac{1}{q^2}\left(\frac{B - u^2}{B}\right)(q^4 U)_{12} - \frac{1}{q^2}\left(\frac{B - u^2}{B}\right)(q^4 U)_1 \cot\theta \\
& - 4\left(\frac{B - u^2}{B}\right)(q^2 \sigma_o)_1 \\
& + 2e^{2(\beta-\gamma)}[-1 - (3\gamma_2 - \beta_2)\cot\theta - \gamma_{22} + \beta_{22} \\
& + {\beta_2}^2 + 2\gamma_2(\gamma_2 - \beta_2)]
\end{aligned} \tag{3.17}$$

Since

$$\frac{e^{2(\gamma+\beta)}}{\sin^2\theta} \bar{R}_{33} = -q^2 e^{2\beta}\lambda \tag{3.18}$$

then (2.6) becomes

$$-q^2e^{2\beta}\lambda = 2\left(\frac{B-u^2}{B}\right)q(q\gamma_o)_1 + 2\left(\frac{B-u^2}{B}\right)\left[\frac{2u}{B-u^2}+\frac{2}{B}\left(\frac{B+u^2}{B-u^2}\right)q\right]\frac{(q^4\gamma_1)_1}{q}$$
$$-\frac{4u}{B-u^2}\left(\frac{B-u^2}{B}\right)q^2\gamma_1 + \left(\frac{B-u^2}{B}\right)^2[(1-q\gamma_1)(\phi^3V)]_1$$
$$-q\left(\frac{B-u^2}{B}\right)(U_2+3U\cot\theta) + 2q\left(\frac{B-u^2}{B}\right)\gamma_2 U$$
$$\times q^2\left(\frac{B-u^2}{B}\right)U_1\cot\theta + q^2\left(\frac{B-u^2}{B}\right)\gamma_2 U_1 + 2q^2\left(\frac{B-u^2}{B}\right)\gamma_{12}U$$
$$+q^2\left(\frac{B-u^2}{B}\right)\gamma_1(U_2+U\cot\theta) - 2\left(\frac{B-u^2}{B}\right)(q^2\sigma_o)_1$$
$$+e^{2(\beta-\gamma)}[-1-(3\gamma_2-2\beta_2)\cot\theta-\gamma_{22}+2\gamma_2(\gamma_2-\beta_2)] \quad (3.19)$$

Since $\bar{R}_{o2} = \lambda q^2 U e^{2\gamma}$, then (2.7) becomes

$$\lambda q^2Ue^{2\gamma} = \beta_{2o} + q^3\sigma_o\left(\frac{\beta_2}{q_2}\right)_1 - q\sigma_o\gamma_{21} - \gamma_{20} + 2\gamma_2(q\sigma_o\gamma_1+\gamma_o)$$
$$-2(q\sigma_o\gamma_1+\gamma_o)\cot\theta - U(\beta_{22}+2\beta_2{}^2+2\beta_2\gamma_2+\beta_2\cot\theta)$$
$$+\left(\frac{B-u^2}{B}\right)q^2e^{2(\gamma-\beta)}\left[\frac{U_{o1}}{2}+\frac{(q^3\sigma_oU_1)_1}{2q^2}+\frac{2(q^2\sigma_o)_1U}{q^2}+(\gamma_oU)_1\right.$$
$$+\frac{(q^2\gamma_o)_1U}{q^2}+\frac{2(q^2\sigma_o\gamma_1)_1U}{q}+\frac{(q^2\sigma_o)_1\gamma_1(q^2U)_1}{q^3}+\frac{3U(q^2U_2)_1}{2q^2}$$
$$+U_1U_2+(\gamma_1U)_2U+\frac{(q^4U^2)_1\cot\theta}{4q^4}+\frac{(\gamma_2q^2U^2)_1}{q^2}+\gamma_1U^2\cot\theta$$
$$-q\sigma_o\beta_1U_1-\beta_oU_1-\beta_2UU_1+\left(\frac{B-u^2}{B}\right)\left(-\frac{(U\phi^3V)_1}{q^2}\right.$$
$$\left.\left.-\frac{(q^2U_1)_1(\phi^3V)}{2q^3}+\frac{(q\gamma_1U\phi^3V)_1}{q^2}+\frac{\beta_1U_1(\phi^3V)}{q}\right)\right]$$
$$-\frac{1}{2}\left(\frac{B-u^2}{B}\right)^2q^4e^{4(\gamma-\beta)}UU_1{}^2$$
$$+\left(\frac{B-u^2}{B}\right)\left[\frac{(\gamma_1-\beta_1)(\phi^3V_2)}{q}-\frac{1}{2}\left(\frac{\phi^3V_2}{q}\right)_1\right] \quad (3.20)$$

Since

$$\bar{R}_{oo} = \lambda\left(\frac{(\phi^3 V)e^{2\beta}}{q} - q^2U^2e^{2\gamma}\right)$$

then (2.8) becomes

$$\lambda\left(\frac{(\phi^3 V)e^{2\beta}}{q} - q^2U^2e^{2\gamma}\right) = \frac{4}{B}\left(\frac{B+2uq}{B-u^2}\right) - 4q\sigma_o{}^2\beta_1 - 4\sigma_o\beta_o$$
$$+ \left(\frac{B-u^2}{B}\right)\left\{-\frac{(\phi^3 V)_o}{q^2} + \frac{2(q^2\sigma_o)_1(\phi^3 V)}{q^3} + \frac{2(\phi^3 V)(q\beta_o)_1}{q^2} + \frac{2(\phi^3 V)(q^3\sigma_o\beta_1)_1}{q^3}\right.$$
$$+ \frac{U}{2}\left(\frac{\phi^3 V}{q}\right)_1 \cot\theta + \frac{U_2}{2}\left(\frac{\phi^3 V}{q}\right)_1 - \frac{(q^4U)_1(\phi^3 V_2)}{2q^5} + \frac{U(\phi^3 V_2)_1}{q}$$
$$+ \frac{(q^2\beta_2 U)_1(\phi^3 V)}{q^3} + \frac{(\beta_1 U\phi^3 V)_2}{q} + \frac{\beta_1 U(\phi^3 V)}{q}\cot\theta$$
$$- \frac{2\gamma_1 U(\phi^3 V_2)}{q} - \left(\frac{B-u^2}{B}\right)\left(\frac{(\phi^3 V)(\phi^3 V)_{11}}{2q^2} + \frac{\beta_1(\phi^3 V)(\phi^3 V)_1}{q^2}\right.$$
$$\left. + \frac{(q\beta_1)_1(\phi^3 V)^2}{q^3}\right) + q^2e^{2(\gamma-\beta)}\left[-UU_{o1} - \frac{2(q^2\sigma_o)_1U^2}{q^2}\right.$$
$$- \frac{U(q^3\sigma_o U_1)_1}{q^2} - 2\gamma_{o1}U^2 - \frac{2U(Uq^3\sigma_o\gamma_1)_1}{q^2} - \frac{2U(qU)_1\gamma_o}{q}$$
$$- 2U(UU_1)_2 - \frac{(U^3)_2}{q} - \frac{(q^3U^3)_1}{3q^3}\cot\theta - \frac{(q^2\gamma_2U^3)_1}{q^2} - (\gamma_1 U)_2U^2$$
$$- \gamma_1 U^3\cot\theta + \left(\frac{B-u^2}{B}\right)\left(\frac{(q^4U_1)_1(\phi^3 V)U}{q^5} + \frac{U^2(\phi^3 V)_1}{q^2} + \frac{U_1{}^2(\phi^3 V)}{q^2}\right.$$
$$\left.\left.\left. + \frac{(\gamma_1U^2q\phi^3 V)_1}{q^2} - \frac{2\beta_1UU_1(\phi^3 V)}{q}\right) + 2(q\sigma_o\beta_1 + \beta_o)UU_1 + 2\beta_2U^2U_1\right]\right\}$$
$$- 2q\sigma_o(\beta_1U)_2 - 2(\beta_oU)_2 + 2q\sigma_o(\gamma_1U)_2 + 2(\gamma_oU)_2 + q\sigma_oU_{12}$$
$$+ U_{o2} + \tfrac{1}{2}(U^2)_{22} + (\gamma_2U^2)_2 - 2\beta_2UU_2 + (2\beta_2{}^2 - 2\beta_2\gamma_2)U^2$$
$$+ 2(q\sigma_o\gamma_1 + \gamma_o)^2 - \cot\theta[2(q\sigma_o\beta_1 + \beta_o - q\sigma_o\gamma_1 - \gamma_o)U$$
$$- q\sigma_oU_1 - U_o - UU_2 - \gamma_2U^2] + \frac{1}{2}\left(\frac{B-u^2}{B}\right)^2 q^4e^{4(\gamma-\beta)}U^2U_1{}^2$$
$$- \frac{e^{2(\beta-\gamma)}}{2q^3}[\phi^3 V_{22} + 2\beta_{22}(\phi^3 V) + (2\beta_2 - 2\gamma_2 + \cot\theta)(\phi^3 V_2 + 2\beta_2\phi^3 V)] \tag{3.21}$$

And finally, since $\bar{R}_{o1} = \lambda\phi^2 e^{2\beta}$, then (2.11) becomes

$$\lambda e^{2\beta} = \left(\frac{B-u^2}{B}\right)\left(\frac{2(q^2\sigma_o)_{11}}{q} + \frac{(q^2\beta_2 U)_1}{q^2} + \frac{(q^2U_2)_1}{2q^2} + \frac{(q^2U)_1 \cot\theta}{2q^2}\right.$$
$$\left. + 2\beta_{1o} + 2(q\sigma_o\beta_1)_1 + (\gamma_1 U)_2 + \gamma_1 U\cot\theta + 2\gamma_1(\gamma_o + q\sigma_o\gamma_1)\right)$$
$$+ \left(\frac{B-u^2}{B}\right)^2\left[-\frac{(\phi^3 V)_{11}}{2q} - \frac{(q\beta_1\phi^3 V)_1}{q^2} + q^2 e^{2(\gamma-\beta)}\left(\frac{(q^4UU_1)_1}{2q^4}\right.\right.$$
$$\left.\left. + (\gamma_1 - \beta_1)UU_1\right)\right] - \frac{e^{2(\beta-\gamma)}}{q^2}[\beta_{22} + 2\beta_2(\beta_2 - \gamma_2) + \beta_2\cot\theta] \quad (3.22)$$

Equations (3.14), (3.17), (3.19)–(3.22) are the required de Sitter space field equations.

4. SOLUTION BY SERIES EXPANSION

The complex of equations (3.14), (3.15), and (3.17) for β, U, and V, respectively, can be solved if we assume a power series expansion in $1/q$ for γ of the form

$$\gamma = \frac{f(u,\theta)}{q} + \frac{d(u,\theta)}{q^2} + \frac{g(u,\theta)}{q^3} + \frac{k(u,\theta)}{q^4} + \theta(q^{-5}) \quad (4.1)$$

The d term is dropped since it gives rise to a log term in the solution for U. The solution for (3.14) then becomes

$$\beta = H(u,\theta) - \frac{(1/4)f^2}{q^2} - \frac{(3/4)fg}{q^4} - \frac{(4/5)fk}{q^5} + \theta(q^{-6}) \quad (4.2)$$

where H is an integration constant. The H term is also dropped since it gives rise to a log term in the solution for V. However, Bondi et al. show that this term can be reduced to zero by a suitable coordinate transformation. Using (4.1) and (4.2) in (3.15), we obtain

$$\left(\frac{B-u^2}{B}\right)U = L - (f_2 + 2f\cot\theta)\frac{1}{q^2} + (2N + 3ff_2 + 4f^2\cot\theta)\frac{1}{q^3}$$
$$- \frac{1}{4}(12fN + 13f^2f_2 + 14f^3\cot\theta - 6g_2 - 12g\cot\theta)\frac{1}{q^4} + \theta(q^{-5}) \quad (4.3)$$

where L and N are integration constants.

Using (4.3) the solution for V from (3.16) is

$$\left(\frac{B-u^2}{B}\right)^2(\phi^3 V) = \frac{1}{3}\left(\frac{12(B+u^2)}{B^2} - \lambda\right)q^3 + \left(\frac{4u}{B} + L_2 + L\cot\theta\right)q^2$$
$$+ (1 + \tfrac{1}{2}f^2\lambda)q - 2M - [N_2 + N\cot\theta - f_2{}^2 - 4ff_2\cot\theta$$
$$- \tfrac{1}{2}f^2(1 + 8\cot^2\theta) + \lambda(\tfrac{3}{2}fg - \tfrac{1}{8}f^4)]\frac{1}{q}$$
$$- \tfrac{1}{2}[g_{22} + 3g_2 - 2g + 6N(f_2 + 2f\cot\theta) + 9ff_2{}^2$$
$$+ \tfrac{1}{2}ff_{22} + \tfrac{51}{2}f_2 f^2\cot\theta + 16f^3\cot^2\theta - \tfrac{1}{3}f^3$$
$$+ \tfrac{8}{5}fk\lambda]\frac{1}{q^2} + \theta(q^{-3}) \tag{4.4}$$

If we replace the parameter $f \to C - c^3/6$, the solution represented by equations (4.1), (4.2), (4.3), and (4.4) will then reduce to the asymptotic flat-space results in the limit $B \to \infty$ and $\lambda \to 0$ provided we put $L = 0$. In asymptotic flat space, the constant of integration L in the equation for U [equation (4.3)] must vanish in order to preserve the signature of the metric. In flat space $V \sim r$ and $U \sim L$. Thus

$$g_{oo} = \frac{Ve^{2\beta}}{r} - U^2 re^{2\gamma} \tag{4.5}$$

would eventually change sign for large enough r. But in asymptotic de Sitter space, $\phi^3 V \sim \theta(q^3)$ and $U \sim L$, so that $\bar{g}_{oo}$ does not change sign for large q. Thus we cannot a priori set $L = 0$ as in the flat-space case. The solution represented by γ, β, U, and V [equations (4.1)–(4.4)] is then substituted into the time development equation (3.19). This yields the following conditions on the expansion:

$$\tfrac{2}{3}f\lambda = L_2 - L\cot\theta \tag{4.6}$$

and

$$4\left(\frac{B-u^2}{B}\right)g_0 = 2fM - N_2 + N\cot\theta + \tfrac{4}{3}k\lambda - 6gL_2 - 4g_2L - 6gL\cot\theta \tag{4.7}$$

With these conditions, the form of γ is preserved and the development of the system is fully determined from initial conditions provided the functions f, k, N, M, L are known. The consistency field equation (3.22) is trivially satisfied to $\theta(q^{-4})$ by this solution for γ, β, U, and V.

This solution, along with the constraints (4.6) and (4.7), reduces the supplementary field equations (3.20) and (3.21) to inverse-square form.

Setting these $\theta(q^{-2})$ terms equal to zero yields, respectively, the following relations:

$$\begin{aligned}\left(\frac{B-u^2}{B}\right)M_o = &-\left(\frac{B-u^2}{B}\right)^2 f_o^2 + \left(\frac{B-u^2}{B}\right)[\tfrac{1}{2}(f_{22o} + 3f_{2o}\cot\theta - 2f_o)\\ &- ff_oL\cot\theta - 2f_of_2L - 2ff_oL_2] + \tfrac{1}{2}\lambda(N_2 + N\cot\theta)\\ &+ \tfrac{1}{2}\lambda g(L_2 - L\cot\theta) - \tfrac{1}{2}L(2M_2 + 3M\cot\theta - f_{222}\\ &- 2f_{22}\cot\theta + 2f_2 + 4f_2\cot^2\theta + 2f\cot\theta + 2f\cot^3\theta)\\ &- \tfrac{1}{2}L_2(3M - 4f_{22} - 10f_2\cot\theta - 2f\cot^2\theta + 4f)\\ &- \tfrac{1}{2}L^2(-12ff_{22} + 2f_2^2 + \tfrac{9}{2}ff_2\cot\theta + f^2 + \tfrac{25}{4}f^2\cot^2\theta)\\ &- \tfrac{1}{2}LL_2(8N + \tfrac{19}{2}ff_2 + \tfrac{13}{2}f^2\cot\theta) - \tfrac{5}{8}L_2^2f^2 + \tfrac{3}{2}L_{22}f_2 \end{aligned} \quad (4.8)$$

$$\begin{aligned}-3\left(\frac{B-u^2}{B}\right)N_o = &M_2 + \left(\frac{B-u^2}{B}\right)(3ff_{2o} + 4ff_o\cot\theta + f_of_2)\\ &- \lambda(g_2 + g\cot\theta) + L_2(2N + \tfrac{15}{4}ff_2 + \tfrac{5}{2}f^2\cot\theta)\\ &+ L(3N_2 + 3N\cot\theta + 3ff_{22} + f_2^2 + \tfrac{9}{4}ff_2\cot\theta\\ &- 3f^2 - \tfrac{9}{2}f^2\cot^2\theta)\end{aligned} \quad (4.9)$$

where (4.9) was used in equation (4.8) in order to remove the N_o dependence from Eq. (4.8) Thus the time development of M and N are known provided the functions g, L, M, N are given for one value of u and f is given as a function of u and θ. Thus we are at the peculiar point that we need to know f, k, and L as functions of u and θ. The reason we need to know L is that the constraint between f and L from equation (4.6) still leaves L unknown up to an arbitrary function of u. We can, however, obtain an equation for k_o provided we carry out the expansion of the time development field equation (3.19) to $\theta(q^{-3})$. But this would still leave L arbitrary. In the next section we show how we can make sense of these parameters by considering the asymptotic de Sitter space limit.

5. ASYMPTOTIC DE SITTER SPACE

An alternate approach to this problem is to require that the solution have a definite form in the limit of $q \to \infty$. This would be, of course, the "asymptotic" de Sitter space form of the metric

$$\bar{g}_{oo} \to \phi^2 = \frac{4u^2}{(B-u^2)^2}q^2 + \frac{4uB}{(B-u^2)^2}q + \frac{B^2}{(B-u^2)^2} \quad (5.1)$$

But from the solution (4.1)–(4.4) and the form of $\bar{g}_{oo}$ from the metric (3.3), we have

$$\bar{g}_{oo} = \frac{B^2}{(B-u^2)^2}\left\{\left(\frac{4(B+u^2)}{B^2} - \frac{\lambda}{3} - L^2\right)q^2 + \left(\frac{4u}{B} + L_2 + L\cot\theta - 2fL^2\right)q + \left[1 + \frac{2}{3}f^2\lambda - \frac{2f^2(B+u^2)}{B^2} - 2f^2L^2 + 2L(f_2 + f\cot\theta)\right] + \theta\left(\frac{1}{q}\right)\right\} \tag{5.2}$$

Thus in the limit of large q, we must have

$$\frac{4u^2}{(B-u^2)^2} = \frac{B^2}{(B-u^2)^2}\left(\frac{4(B+u^2)}{B^2} - \frac{\lambda}{3} - L^2\right) \tag{5.3}$$

or

$$\frac{\lambda}{3} - \frac{4}{B} = L^2 \tag{5.4}$$

and similarly,

$$2fL^2 = L_2 + L\cot\theta \tag{5.5}$$

$$\tfrac{2}{3}f^2\lambda - \frac{2f^2(B+u^2)}{B^2} - 2f^2L^2 + 2L(f_2 + f\cot\theta) = 0 \tag{5.6}$$

Equation (5.4) immediately implies that L is independent of the coordinates and is therefore an invariant. But this implies with equation (5.5) that either $L = 0$ or $f \propto \cot\theta$. The latter case does not have the correct regularity properties (Bondi et al.) for γ as $\theta \to 0$. Thus

$$L = 0 \qquad \lambda = 12/B \tag{5.7}$$

Finally, equation (5.6) implies that either $f = 0$ or $B = u^2/3$. Previously we ruled out $f \propto \cot\theta$ on the basis of regularity for γ. But from equation (4.6) we must now conclude that either $f = 0$ or $\lambda = 0$. The consistent choice is

$$f = 0 \tag{5.8}$$

Thus by comparison with the asymptotic flat-space case, the character of the solution is very different in asymptotic de Sitter space. Indeed, the mass aspect [equation (4.8)] now depends on a different "news" function that becomes, due to (4.7), (5.7), and (5.8),

$$M_o = \frac{6}{(B-u^2)}(N_2 + N\cot\theta) = -\frac{24}{B}g_o + \frac{12}{B-u^2}N\cot\theta + \frac{96}{B(B-u^2)}k \tag{5.9}$$

Similarly for N_o, we find

$$N_o = -\frac{BM_2}{3(B - u^2)} + \frac{4}{(B - u^2)}(g_2 + g \cot \theta) \tag{5.10}$$

On the other hand, the radiation condition[2] in asymptotic flat space expressed in (q, u) coordinates becomes

$$\frac{\partial}{\partial r}(r\gamma') = \frac{\partial}{\partial q}(q\gamma)\left(\frac{B + 2uq}{B}\right) - \frac{2uq\gamma}{B} \tag{5.11}$$

where γ' is a function of $1/r$; so in the limit of $q \to \infty$, the right-hand side will not vanish *unless* $f = 0$. Thus the condition for radiation in q coordinates seems to be satisfied if $f = 0$, and the description of mass or mass loss now depends consistently on g_o in the supplementary conditions, i.e., equation (5.9). This unusual behavior for γ is not totally surprising since here the asymptotic limit cannot be described by an empty-space axially symmetric static metric, and therefore related to Weyl's form, from which Bondi et al. could identify the mass function.

6. CONCLUSION

Provided that equation (5.11) is also a correct description for radiation in asymptotic de Sitter space, we conclude that radiation will occur in an asymptotic de Sitter space but that the nature of the solution requires that the $1/q$ dependence of the solution for γ begin with the $\theta(q^{-3})$ term. This implies that in the limit $B \to \infty$, we do not obtain the radiative flat-space results. That is, the requirement for radiation in asymptotic de Sitter space seems to be more restrictive than in asymptotic flat space.

On the other hand, equation (5.11) may not be an adequate description for radiation in a de Sitter space, in which case our solution is an unusual curiosity but does not represent radiation. Alternately, we could conclude that de Sitter space is devoid of physical meaning.

We now are ready to return to the argument following equation (5.7) in which we concluded that $f = 0$. This choice was, however, only consistent with a solution in asymptotic de Sitter space. Instead, if we had chosen $\lambda = 0$, then $B \to \infty$, which would then automatically restrict our solution to

[2] The form of the metric in de Sitter space does not seem to conform to the more general "Sommerfeld radiation condition" described by A. Trautman, "Gravitational Waves and Radiation," presented at the London Conference on Theories of Gravitation, 1965 (unpublished), since the limiting metric is not Minkowskiian. We have therefore attempted to describe radiation in the more general de Sitter space by carrying over directly the form used by Bondi et al.

flat space. Thus we are left with the unusual consequences of probable radiation in de Sitter space or no de Sitter space solution at all.

ACKNOWLEDGMENT

I wish to thank Dr. Tom Morgan for suggesting this problem and for his continued support during its investigation.

REFERENCES

Aghassi, J. J., Roman, P., and Santille, R. M. (1970). *Journal of Mathematical Physics*, **11**, 2297.

Bondi, H., Van der Burg, M. G. H., and Metzner, A. W. K. (1962). *Proceedings of the Royal Society of London*, **269**, 21.

Eisenhart, L. P. (1926). *Riemannian Geometry*, pp. 89–90. Princeton University Press, Princeton.

Gürsey, F. (1964). "Introduction to the de Sitter Group," in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, by F. Gürsey, ed., pp. 365–389. Gordon and Breach, New York.

Hawking, S. W. (1968). *Journal of Mathematical Physics*, **9**, 598.

Hsu, J. P. (1977). "Gravitation as the gauge theory with maximum four-dimensional symmetry," preprint, Space Sciences Laboratory, Marshall Space Flight Center.

Lee, D. L., and Lightman, A. P. (1973). *Physical Review D*, **7**, 3578.

Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*, Chap. 32. Freeman, San Francisco.

Pirani, F. A. E. (1964). In *Lectures on General Relativity*, Brandeis Summer Institute in Theoretical Physics, Vol. 1, A. Trautman et al., eds., p. 368. Prentice-Hall, Englewood Cliffs.

Sachs, R. R. (1962). *Proceedings of the Royal Society of London*, **270**, 103.

Shapiro, I. I., Counselman, C. C. III, and King, R. W. (1976). *Physical Review Letters*, **36**, 555.

Tait, W., and Cornwell, J. F. (1971). *Journal of Mathematical Physics*, **12**, 1651.

Truesdell, C. (1953). *Zeitschrift für Angewandte Mathematique und Mechanik*, **33**, 345.

Weinberg, S. (1972). *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, p. 155. Wiley, New York.

Will, C. M. (1974a). *Experimental Tests of Gravitation Theories* (Unpublished); (1974b). "Gravitation Theory," *Scientific American*, **231**, No. 5, 24; (1974c). "The Theoretical Tools of Experimental Gravitation," in *Proceedings of the International School of Physics, Enrico Fermi Course LVI, Experimental Gravitation*, p. 21. Academic, New York.

Williams, J. G., Dicke, R. H., Bender, P. L., Alley, C. O., Carter, W. E., Currie, D. G., Eckhardt, D. H., Faller, J. E., Kaula, W. M., Mulholland, J. D., Plotkin, H. H., Poultney, S. K., Shelus, P. J., Silverberg, E. C., Sinclair, W. S., Slade, M. A., and Wilkinson, D. T. (1976). *Physical Review Letters*, **36**, 551.